Injective Coloring of Interval graphs

Rumki Ghosh, B. S. Panda

Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi 110016, INDIA

Abstract

An injective k-coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \ldots, k\}$ such that for every pair of vertices u and v having a common neighbor, $f(u) \neq f(v)$. The injective chromatic number $\chi_i(G)$ of a graph G is the minimum k for which G admits an injective k-coloring. Given a graph G and a positive integer k, DECIDE INJECTIVE COLORING PROBLEM is to decide whether G admits an injective k-coloring. We prove that the injective chromatic number of an interval graph is either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. We also characterize the interval graphs having $\chi_i(G) = \Delta(G)$ and $\chi_i(G) = \Delta(G) + 1$. As a consequence of this characterization, we obtain a linear-time algorithm to find the injective chromatic number of an interval graph.

Keywords: Coloring, Injective Coloring, Interval Graph, Polynomial-time Algorithm.

---

1. Introduction

A vertex k-coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \ldots, k\}$. An injective k-coloring of a graph G is a k-coloring of G such that no two vertices having a common neighbor receive the same color; that is, the restriction of the k-coloring to the neighborhood of any vertex is injective. The injective chromatic number $\chi_i(G)$ of a graph G is the minimum k for which G admits an injective k-coloring. The INJECTIVE COLORING PROBLEM is to find an injective coloring of G with $\chi_i(G)$ colors. The decision version of the INJECTIVE COLORING PROBLEM is as stated follows:

<u>DECIDE INJECTIVE COLORING PROBLEM</u>

Instance: A graph $G = (V, E)$ and a positive integer k.

Question: Does G have an injective k-coloring?

The concept of injective coloring was introduced in 2002 by Hahn et al. [1]. It was originated from complexity theory of random access machines and it has application in the theory of error correcting codes [1]. Hahn et al. [1] showed that $\Delta(G) \leq \chi_i(G) \leq \Delta(G)(\Delta(G) - 1) + 1$, where $\Delta(G)$ is the maximum degree of G and characterized the d-regular graphs that achieve the lower bound and all the graphs that achieve the upper bound. They also proved that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for general graphs. Hell et al. [2] proved that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for chordal graphs by showing the NP-completeness for split graphs. They [2] provided a polynomial time algorithm for the injective chromatic number of power chordal graphs. The injective chromatic number of a tree can be computed in polynomial time [2]. Panda et al. [3] showed that the injective chromatic number of proper interval graphs, threshold graphs, and $K_{1,3}$-free split graphs can be determined in linear time. In this paper, we study the injective chromatic number of an interval graph. Though an $O(nm)$ time algorithm for computing $\chi_i(G)$ for an interval graph follows from [2] as interval graphs are power chordal graphs, finding an $O(n+m)$ time algorithm for computing the injective chromatic number of interval graphs is still open. The main contributions of the paper are summarized below.

1. For an interval graph G, it is proved that $\Delta(G) \leq \chi_i(G) \leq \Delta(G) + 1$.
2. Interval graphs satisfying $\chi_i(G) = \Delta(G)$ and $\chi_i(G) = \Delta(G) + 1$ are characterized.
3. The characterization so obtained leads to a linear-time algorithm to find the injective chromatic number of an interval graph. This improves the result of Panda et al. [3] for proper interval graphs.

Email addresses: maz218120@maths.iitd.ac.in (Rumki Ghosh), bspanda@maths.iitd.ac.in (B. S. Panda)

2. Main Results

2.1. Bounds for the Injective Chromatic Number of an Interval Graph

We first show that $\Delta(G) \leq \chi_i(G) \leq \Delta(G)+1$ for an interval graph G. Let $G=(V,E)$ be an interval graph with an interval ordering $\sigma=(v_1,v_2,\ldots,v_n)$. Let $S_{Max}=\{v_{r_1},v_{r_2},...,v_{r_k}\}$ be the set of all maximum degree vertices such that $r_1<r_2<\cdots<r_k$ with respect to σ. For each $v_{r_i}\in S_{Max}$, $v_{l_i}=min(N[v_{r_i}])$ denote the minimum neighbor of v_{r_i} and $v_{f_i}=max(N[v_{r_i}])$ denote the maximum neighbor of v_{r_i} with respect to σ. Let $V_{Pendant}$ and $V_{Lpendant}$ be defined as follows.

- $V_{Pendant}=\{v_{p_1},v_{p_2},\ldots,v_{p_a}\}$ be the set of all pendant vertices in G such that for all $j=1,2,\ldots,a$, $v_{p_j}\in N[v_{r_i}]$ for some $v_{r_i}\in S_{Max}$ where $p_1<p_2<\ldots<p_a$ with respect to σ.
- $V_{Lpendant}=\{v_{q_1},v_{q_2},\ldots,v_{q_b}\}$ be the set of all vertices which are not pendant in G but pendant in $G[N[v_{r_i}]]$ for some $v_{r_i}\in S_{Max}$ where $q_1<q_2<\ldots<q_b$ with respect to σ.

GREEDY INJECTIVE COLORING ALGORITHM: Given an ordering $\alpha=(v_1,v_2,\ldots,v_n)$ of vertices of $G=(V,E)$, the greedy injective coloring algorithm assigns each vertex v_i the first available color that is not used by any vertex $v_j, j<i$ that has a common neighbor with v_i. The colors which are assigned to the vertices $v_j, j<i$ that have a common neighbor with v_i are said to be forbidden for v_i.

We will use a suitable ordering of the vertices of an interval graph and apply the greedy injective coloring to obtain optimal injective coloring.

Theorem 2.1. *If G is an interval graph, then $\Delta(G)\leq\chi_i(G)\leq\Delta(G)+1$.*

2.2. Characterizations of Interval Graphs achieving the Bounds

We now characterize interval graphs with $\chi_i(G)=\Delta(G)$ and interval graphs with $\chi_i(G)=\Delta(G)+1$. We define three types of maximum degree vertices in G based on their neighbors and their positions in the interval ordering of G.

Definition 2.1. *Let G be an interval graph and $\sigma=(v_1,v_2,\ldots,v_n)$ be an interval ordering of G. Let $S_{Max}=\{v_{r_1},v_{r_2},...,v_{r_k}\}$ be the set of all maximum degree vertices such that $r_1<r_2<\cdots<r_k$ with respect to σ.*

1. *L-vertex: A maximum degree vertex $v_{r_i}\in S_{Max}$ is called an L-vertex if v_{l_i}, the minimum neighbor of v_{r_i}, is the only neighbor in $V_{LPendant}$ and no neighbor of v_{r_i} is in $V_{Pendant}$.*
2. *R-vertex: A maximum degree vertex $v_{r_i}\in S_{Max}$ is called an R-vertex if v_{f_i}, the maximum neighbor of v_{r_i}, is the only neighbor in $V_{LPendant}$ and no neighbor of v_{r_i} is in $V_{Pendant}$.*
3. *LR-vertex: A maximum degree vertex $v_{r_i}\in S_{Max}$ is called an LR-vertex if v_{l_i} and v_{f_i}, the minimum neighbor and the maximum neighbor, are the only two neighbors in $V_{LPendant}$ and no neighbor of v_{r_i} is in $V_{Pendant}$.*

The L-vertex, R-vertex, and LR-vertex are illustrated in the Figure 1.

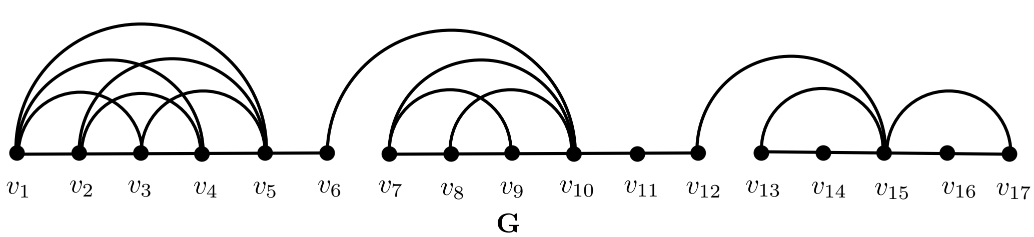

Figure 1: An example of an R- vertex v_5, an LR-vertex v_{10} and an L vertex v_{15} of the interval graph G

Definition 2.2. *Let G be an interval graph with an interval ordering $\sigma=(v_1,v_2,\ldots,v_n)$. Let $S_{Max}=\{v_{r_1},v_{r_2},...,v_{r_k}\}$ be the set of all maximum degree vertices such that $r_1<r_2<\cdots<r_k$ with respect to σ. G is said to be a* TYPE-1 *interval graph with respect to σ if it satisfies at least one of the following conditions:*

C1: There exists a vertex $v_{r_i} \in S_{Max}$ such that it has no pendant neighbor in $V_{Pendant}$ and it has no pendant neighbor in $V_{Lpendant}$.
C2: There exist two vertices, an R-vertex $v_{r_i} \in S_{Max}$ and an L-vertex $v_{r_j} \in S_{Max}$ such that $v_{f_i} = v_{l_j}$.
C3: There exists $\alpha + 2, \alpha \geq 1$ vertices consisting of an R-vertex $v_{r_i} \in S_{Max}$, α number of LR-vertices $v_{r_{j_1}}, v_{r_{j_2}}, \ldots, v_{r_{j_\alpha}} \in S_{Max}$ and an L-vertex $v_{r_p} \in S_{Max}$ such that $v_{f_i} = v_{l_{j_1}}, v_{f_{j_1}} = v_{l_{j_2}}, \ldots, v_{f_{j_{\alpha-1}}} = v_{l_{j_\alpha}}$, and $v_{f_{j_\alpha}} = v_{l_p}$.
Otherwise, G is said to be a TYPE-2 *interval graph with respect to σ.*

TYPE-1 and TYPE-2 graphs are illustrated in Figure 2.

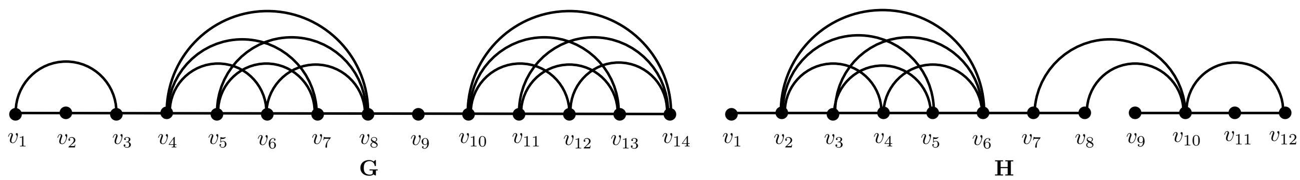

Figure 2: An example of a TYPE-1 interval graph G and a TYPE-2 interval graph H

Theorem 2.2. *If G is a* TYPE-1 *interval graph, then $\chi_i(G) = \Delta(G) + 1$.*

An example of an injective coloring of an interval graph G with $\chi_i(G) = \Delta(G) + 1$ is illustrated in Figure $3(a)$.

Theorem 2.3. *If G is a* TYPE-2 *interval graph with respect to σ, then $\chi_i(G) = \Delta(G)$.*

An example of an injective coloring of an interval graph G with $\chi_i(G) = \Delta(G)$ is illustrated in Figure $3(b)$.

(a) Injective Coloring of a TYPE-1 Interval Graph

(b) Injective Coloring of a TYPE-2 Interval Graph

Figure 3: Injective Coloring of two types of Interval Graphs

2.3. Linear time Algorithm for Injective Chromatic Number of an Interval Graph

Note that in the proof of the theorems, we have used the greedy injective coloring algorithm using the reverse ordering of an interval order by suitably reordering certain vertices in some cases. To injectively color the vertex v_i, we need to find all v_j with $j > i$ at distance two through the maximum neighbor v_k, maintaining distinct colors of v_k, avoiding recomputation, and since each adjacency list is scanned once, the overall running time is $O(m + n)$. Hence, we have the following theorem.

Theorem 2.4. *An optimal injective coloring of an interval graph G can be computed in $O(n + m)$ time.*

References

[1] G. Hahn, J. Kratochvıl, J. Širáň, and D. Sotteau. On the injective chromatic number of graphs. *Discrete mathematics*, 256(1-2):179–192, 2002.

[2] P. Hell, A. Raspaud, and J. Stacho. On injective colourings of chordal graphs. In *Latin American Symposium on Theoretical Informatics, Springer*, pages 520–530, 2008.

[3] B. S. Panda and Priyamvada. Injective coloring of some subclasses of bipartite graphs and chordal graphs. *Discrete Applied Mathematics*, 291:68–87, 2021.